

A HYPONORMAL WEIGHTED SHIFT ON A DIRECTED TREE WHOSE SQUARE HAS TRIVIAL DOMAIN

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ABSTRACT. It is proved that, up to isomorphism, there are only two directed trees that admit a hyponormal weighted shift with nonzero weights whose square has trivial domain. These are precisely those enumerable (i.e., countably infinite) directed trees, one with root, the other without, whose every vertex has an enumerable set of successors. An example of a nonzero hyponormal composition operator in an L^2 -space whose square has trivial domain is established.

1. INTRODUCTION

In recent papers [4, 5] a question of subnormality of unbounded weighted shifts on directed trees has been investigated. A criterion for subnormality of such operators whose C^∞ -vectors are dense in the underlying Hilbert space has been established (cf. [4, Theorem 5.1.1]). It has been written in terms of consistent systems of Borel probability measures. The assumption that the operator in question has a dense set of C^∞ -vectors diminishes the class of weighted shifts on directed trees to which this criterion can be applied (note that the set of all C^∞ -vectors of a classical, unilateral, or bilateral weighted shift is always dense in the underlying Hilbert space). Unfortunately, there is no efficient criterion for subnormality of unbounded Hilbert space operators that have a small set of C^∞ -vectors. The known characterizations of subnormality require the existence of additional objects (like semispectral measures, elementary spectral measures, or sequences of unbounded operators) that have to satisfy appropriate, more or less complicated, conditions (cf. [3, 8, 22, 23]). Among subnormal operators having a small set of C^∞ -vectors, the symmetric ones, which are always subnormal (cf. [1, Theorem 1 in Appendix I.2]), play an essential role. According to [15] (see also [6]) there are closed symmetric operators whose squares have trivial domain. Unfortunately, symmetric weighted shifts on directed trees are automatically bounded; the same is true for formally normal weighted shifts on directed trees (cf. [11, Proposition 3.1]).

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The above discussion leads to the following problem.

Question. Does there exist a subnormal weighted shift on a directed tree with nonzero weights whose square has trivial domain?

At present, this question remains unanswered. However, as shown in Theorem 3.1, there are injective hyponormal weighted shifts on directed trees with nonzero weights whose squares have trivial domain. What is more, the only directed trees admitting a densely defined weighted shift with nonzero weights whose square has trivial domain are those enumerable directed trees whose every vertex has enumerable set of successors (children). We conclude the paper by showing that there exists a nonzero hyponormal composition operator in an L^2 -space whose square has trivial domain (cf. Theorem 4.3).

2. PRELIMINARIES

In what follows, \mathbb{C} stands for the set of all complex numbers. Let A be an operator in a complex Hilbert space \mathcal{H} (all operators considered in this paper are linear). Denote by $\mathcal{D}(A)$ and A^* the domain and the adjoint of A (in case it exists). A closed densely defined operator N in \mathcal{H} is called *normal* if $N^*N = NN^*$. A densely defined operator S in \mathcal{H} is said to be *subnormal* if there exists a complex Hilbert space \mathcal{K} and a normal operator N in \mathcal{K} such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Sh = Nh$ for all $h \in \mathcal{D}(S)$. Finally, a densely defined operator S in \mathcal{H} is called *hyponormal* if $\mathcal{D}(S) \subseteq \mathcal{D}(S^*)$ and $\|S^*f\| \leq \|Sf\|$ for all $f \in \mathcal{D}(S)$. It is well-known that subnormal operators are hyponormal (but not conversely) and that hyponormal operators are closable and their closures are hyponormal (subnormal operators have an analogous property). We refer the reader to [2, 24] for basic facts on unbounded operators, [7, 18–21] for the foundations of the theory of (bounded and unbounded) subnormal operators and [12–14, 16, 17] for elements of the theory of unbounded hyponormal operators.

Let $\mathcal{T} = (V, E)$ be a directed tree (V and E always stand for the sets of vertices and edges of \mathcal{T} , respectively). If \mathcal{T} has a root, which will always be denoted by **root**, then we write $V^\circ := V \setminus \{\text{root}\}$; otherwise, we put $V^\circ = V$. Set $\text{Chi}(u) = \{v \in V : (u, v) \in E\}$ for $u \in V$. If for a given vertex $u \in V$ there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then we denote it by $\text{par}(u)$. The correspondence $u \mapsto \text{par}(u)$ is a partial function from V to V . For an integer $n \geq 1$, the n -fold composition of the partial function par with itself will be denoted by par^n . Let par^0 stand for the identity map on V . We call \mathcal{T} *leafless* if $V = \{u \in V : \text{Chi}(u) \neq \emptyset\}$. If $W \subseteq V$, we put $\text{Chi}(W) = \bigcup_{v \in W} \text{Chi}(v)$ and $\text{Des}(W) = \bigcup_{n=0}^{\infty} \text{Chi}^{(n)}(W)$, where $\text{Chi}^{(0)}(W) = W$ and $\text{Chi}^{(n+1)}(W) = \text{Chi}(\text{Chi}^{(n)}(W))$ for all integers $n \geq 0$. For $u \in V$, we set $\text{Chi}^{(n)}(u) = \text{Chi}^{(n)}(\{u\})$ and $\text{Des}(u) = \text{Des}(\{u\})$.

Let $\ell^2(V)$ be the Hilbert space of all square summable complex functions on V equipped with the standard inner product. For $u \in V$, we define $e_u \in \ell^2(V)$ to be the characteristic function of the one point set $\{u\}$. Given $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, we define the operator S_λ in $\ell^2(V)$ by

$$\begin{aligned} \mathcal{D}(S_\lambda) &= \{f \in \ell^2(V) : \lambda_{\mathcal{T}} f \in \ell^2(V)\}, \\ S_\lambda f &= \lambda_{\mathcal{T}} f, \quad f \in \mathcal{D}(S_\lambda), \end{aligned}$$

where $\Lambda_{\mathcal{T}}$ is the map defined on functions $f: V \rightarrow \mathbb{C}$ via

$$(2.1) \quad (\Lambda_{\mathcal{T}}f)(v) = \begin{cases} \lambda_v \cdot f(\text{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \text{root}. \end{cases}$$

S_{λ} is called a *weighted shift* on the directed tree \mathcal{T} with weights $\{\lambda_v\}_{v \in V^\circ}$. Note that any weighted shift S_{λ} on \mathcal{T} is a closed operator. For the foundations of the theory of weighted shifts on directed trees we refer the reader to [9].

Before proving the main result, we recall a characterization of hyponormality of weighted shifts on directed trees with nonzero weights. In view of [4, Proposition 5.2.1], there is no loss of generality in assuming that underlying directed trees are leafless.

Theorem 2.1 ([9, Theorem 5.1.2 and Remark 5.1.5]). *Let S_{λ} be a densely defined weighted shift on a leafless directed tree \mathcal{T} with nonzero weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then S_{λ} is hyponormal if and only if*

$$\sum_{v \in \text{Chi}(u)} \frac{|\lambda_v|^2}{\|S_{\lambda}e_v\|^2} \leq 1, \quad u \in V.$$

3. A HYPONORMAL S_{λ} WITH $\mathcal{D}S_{\lambda}^2 = \{0\}$

We are now in a position to prove our main result.

Theorem 3.1. *Let \mathcal{T} be a directed tree such that $V^\circ \neq \emptyset$. Then the following assertions are equivalent:*

- (i) $\text{card}(\text{Chi}(u)) = \aleph_0$ for every $u \in V$,
- (ii) there exists a family $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of nonzero complex numbers such that S_{λ} is injective and hyponormal, and $\mathcal{D}(S_{\lambda}^2) = \{0\}$,
- (iii) there exists a family $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of nonzero complex numbers such that $\overline{\mathcal{D}(S_{\lambda})} = \ell^2(V)$ and $\mathcal{D}(S_{\lambda}^2) = \{0\}$.

Note that, up to isomorphism, there are only two directed trees, one with root, the other without, that satisfy condition (i) of Theorem 3.1. Hence, by this theorem, each of them admits a hyponormal weighted shift with nonzero weights whose square has trivial domain.

Proof of Theorem 3.1. Fix $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$. We will show that

(†) a complex function f on V belongs to $\mathcal{D}(S_{\lambda}^2)$ if and only if¹

$$(3.1) \quad \sum_{u \in V} \left(1 + \zeta_u^2 + \sum_{v \in \text{Chi}(u)} \zeta_v^2 |\lambda_v|^2 \right) |f(u)|^2 < \infty,$$

where $\zeta_u := \sqrt{\sum_{v \in \text{Chi}(u)} |\lambda_v|^2}$ for $u \in V$.

Indeed, by [9, Proposition 3.1.3], a complex function f on V belongs to $\mathcal{D}(S_{\lambda})$ if and only if $f \in \ell^2(V)$ and $\sum_{u \in V} \zeta_u^2 |f(u)|^2 < \infty$. Hence, a complex function f on V belongs to $\mathcal{D}(S_{\lambda}^2)$ if and only if $\sum_{u \in V} (1 + \zeta_u^2) |f(u)|^2 < \infty$ and

¹ With the convention that $0 \cdot \infty = 0$ and $\sum_{v \in \emptyset} x_v = 0$.

$\sum_{u \in V} \zeta_u^2 |(S_{\lambda} f)(u)|^2 < \infty$. Since, by (2.1) and the identity² $V^{\circ} = \bigsqcup_{u \in V} \text{Chi}(u)$ (cf. [9, Proposition 2.1.2]), the following equalities hold:

$$\begin{aligned} \sum_{u \in V} \zeta_u^2 |(S_{\lambda} f)(u)|^2 &= \sum_{u \in V^{\circ}} \zeta_u^2 |\lambda_u|^2 |f(\text{par}(u))|^2 \\ &= \sum_{u \in V} \sum_{v \in \text{Chi}(u)} \zeta_v^2 |\lambda_v|^2 |f(\text{par}(v))|^2 \\ &= \sum_{u \in V} \left(\sum_{v \in \text{Chi}(u)} \zeta_v^2 |\lambda_v|^2 \right) |f(u)|^2, \quad f \in \mathcal{D}(S_{\lambda}), \end{aligned}$$

we see that a complex function f on V belongs to $\mathcal{D}(S_{\lambda}^2)$ if and only if (3.1) holds.

(i) \Rightarrow (ii) We begin by proving that

(\dagger) for each $(\vartheta, u) \in (0, \infty) \times V$ there exists a family $\{\lambda_v\}_{v \in \text{Des}(u)} \subseteq (0, \infty)$ such that for every $v \in \text{Des}(u)$, the following three conditions hold:

$$(3.2) \quad \lambda_u^2 = \vartheta,$$

$$(3.3) \quad \left(\sum_{w \in \text{Chi}(v)} \lambda_w^2 \right) \lambda_v^2 = 1,$$

$$(3.4) \quad \sum_{w \in \text{Chi}(v)} \lambda_w^4 \leq 1.$$

For this, we first note that

$$(3.5) \quad \text{for each positive real number } r, \text{ there exists a sequence } \{r_n\}_{n=1}^{\infty} \subseteq (0, 1) \text{ such that } \left(\sum_{j=1}^{\infty} r_j \right) r = 1 \text{ and } \sum_{j=1}^{\infty} r_j^2 \leq 1$$

(e.g., $r_j = \frac{1}{rn}$ for $1 \leq j \leq n-1$ and $r_j = \frac{1}{rn2^{j-n+1}}$ for $j \geq n$, where $n \geq 2$ is chosen so that $\frac{1}{r^2n} \leq 1$). Let us fix $u \in V$ and set $X_n = \text{Chi}^{(0)}(u) \sqcup \cdots \sqcup \text{Chi}^{(n)}(u)$ for $n \geq 0$ (the terms of the sum are pairwise disjoint due to [9, (2.1.10)]). Since $\text{Des}(u) = \bigcup_{n=1}^{\infty} X_n$, we can construct the required family inductively. For $n = 1$, we put $\lambda_u = \sqrt{\vartheta}$ and, by using (3.5), we choose a family $\{\lambda_w\}_{w \in \text{Chi}(u)} \subseteq (0, \infty)$ such that $(\sum_{w \in \text{Chi}(u)} \lambda_w^2) \vartheta = 1$ and $\sum_{w \in \text{Chi}(u)} \lambda_w^4 \leq 1$ (this is possible because $\text{Chi}(u)$ is enumerable). Fix $n \geq 1$, and assume that we already have a family $\{\lambda_v\}_{v \in X_n} \subseteq (0, \infty)$ such that (3.2), (3.3) and (3.4) hold for all $v \in X_{n-1}$. Then for every $v \in \text{Chi}^{(n)}(u)$ we can choose (by using (3.5) again) a family $\{\lambda_w\}_{w \in \text{Chi}(v)} \subseteq (0, \infty)$ which satisfies (3.3) and (3.4). In view of [9, (6.1.3)], this gives us the family $\{\lambda_v\}_{v \in \text{Chi}^{(n+1)}(u)}$ such that (3.3) and (3.4) hold for all $v \in X_n$. Now by induction we get a family $\{\lambda_v\}_{v \in \text{Des}(u)} \subseteq (0, \infty)$ which satisfies the requirements of (\dagger).

Our next aim is to construct a family $\lambda = \{\lambda_v\}_{v \in V^{\circ}} \subseteq (0, \infty)$ with the properties mentioned in (ii).

If \mathcal{T} has a root, then applying (\dagger) to $u = \text{root}$ and $\vartheta = 1$, and employing [9, Corollary 2.1.5] we get a family $\lambda = \{\lambda_v\}_{v \in V^{\circ}} \subseteq (0, \infty)$ such that

$$(3.6) \quad \left(\sum_{w \in \text{Chi}(v)} \lambda_w^2 \right) \lambda_v^2 = 1 \text{ and } \sum_{w \in \text{Chi}(v)} \lambda_w^4 \leq 1 \text{ for all } v \in V.$$

(Of course, the number λ_{root} is not a weight of S_{λ} .)

² The symbol \sqcup denotes a pairwise disjoint union of sets.

Suppose now that \mathcal{T} is rootless. It is easily seen that

$$(3.7) \quad \begin{array}{l} \text{for every } r \in (0, 1), \text{ there exists a sequence } \{r_j\}_{j=1}^\infty \subseteq (0, 1) \\ \text{such that } r + \sum_{j=1}^\infty r_j = 2 \text{ and } r^2 + \sum_{j=1}^\infty r_j^2 \leq 1. \end{array}$$

Take $u_1 \in V$ and set $u_2 = \text{par}(u_1)$. By (\ddagger) , there exists a family $\{\lambda_v\}_{v \in \text{Des}(u_1)} \subseteq (0, \infty)$ with $\lambda_{u_1} = \frac{1}{\sqrt{2}}$, which satisfies (3.3) and (3.4) for all $v \in \text{Des}(u_1)$. In the next step we shall construct a new family $\{\lambda_v\}_{v \in \text{Des}(u_2) \setminus \text{Des}(u_1)} \subseteq (0, \infty)$ with $\lambda_{u_2} = \frac{1}{\sqrt{2}}$ such that the extended family $\{\lambda_v\}_{v \in \text{Des}(u_2)}$ satisfies (3.3) and (3.4) for all $v \in \text{Des}(u_2)$. For this, we note that by [4, (2.2.8)],

$$(3.8) \quad \text{Des}(u_2) \setminus \text{Des}(u_1) = \{u_2\} \sqcup \bigsqcup_{u \in \text{Chi}(u_2) \setminus \{u_1\}} \text{Des}(u).$$

We set $\lambda_{u_2} = \frac{1}{\sqrt{2}}$ and, by using (3.7), we choose a family $\{\vartheta_u\}_{u \in \text{Chi}(u_2) \setminus \{u_1\}} \subseteq (0, 1)$ such that

$$(3.9) \quad \left(\sum_{u \in \text{Chi}(u_2) \setminus \{u_1\}} \vartheta_u + \lambda_{u_1}^2 \right) \lambda_{u_2}^2 = 1,$$

$$(3.10) \quad \sum_{u \in \text{Chi}(u_2) \setminus \{u_1\}} \vartheta_u^2 + \lambda_{u_1}^4 \leq 1.$$

Applying (\ddagger) to $u \in \text{Chi}(u_2) \setminus \{u_1\}$ and $\vartheta = \vartheta_u$, we get a family $\{\lambda_v\}_{v \in \text{Des}(u)} \subseteq (0, \infty)$ with $\lambda_u^2 = \vartheta_u$, which satisfies (3.3) and (3.4) for all $v \in \text{Des}(u)$. This, together with (3.9) and (3.10), leads to $(\sum_{u \in \text{Chi}(u_2)} \lambda_u^2) \lambda_{u_2}^2 = 1$ and $\sum_{w \in \text{Chi}(u_2)} \lambda_w^4 \leq 1$. In view of (3.8), our construction is complete. Applying an induction argument (with $\lambda_{u_n} = \frac{1}{\sqrt{2}}$ for $n \geq 2$) and using the fact that $V = \bigcup_{k=0}^\infty \text{Des}(\text{par}^k(u_1))$ (cf. [9, Proposition 2.1.6]), we construct a family $\lambda = \{\lambda_v\}_{v \in V} \subseteq (0, \infty)$ satisfying (3.6).

Since $\text{card}(\text{Chi}(u)) = \aleph_0$ for all $u \in V$, we infer from [9, Propositions 3.1.3 and 3.1.7], (3.6) and (\dagger) that S_λ is injective and densely defined, and $\mathcal{D}(S_\lambda^2) = \{0\}$. It follows from [9, (3.1.4)] and the equality in (3.6) that $e_v \in \mathcal{D}(S_\lambda)$ and $\lambda_v^2 = \|S_\lambda e_v\|^{-2}$ for all $v \in V$, and thus

$$\sum_{v \in \text{Chi}(u)} \frac{\lambda_v^2}{\|S_\lambda e_v\|^2} = \sum_{v \in \text{Chi}(u)} \lambda_v^4 \stackrel{(3.6)}{\leq} 1, \quad u \in V,$$

which together with Theorem 2.1 implies that S_λ is hyponormal.

(ii) \Rightarrow (iii) Evident.

(iii) \Rightarrow (i) Let S_λ be as in (iii). By [9, Proposition 3.1.10], V is countable. Thus each $\text{Chi}(u)$ is countable. Suppose that, contrary to our claim, (i) does not hold. Then there exists $u_0 \in V$ such that $\text{Chi}(u_0)$ is finite. Since S_λ is densely defined, we infer from assertions (iii) and (v) of [9, Proposition 3.1.3] that $\zeta_v < \infty$ for all $v \in V$. Hence

$$1 + \zeta_{u_0}^2 + \sum_{v \in \text{Chi}(u_0)} \zeta_v^2 |\lambda_v|^2 < \infty.$$

This, combined with (\dagger) , implies that $f = e_{u_0} \in \mathcal{D}(S_\lambda^2)$, which contradicts (iii). \square

Remark 3.2. It is worth mentioning that the hyponormal weighted shift S_λ constructed in the proof of implication (i) \Rightarrow (ii) of Theorem 3.1 has the property that

$\mathcal{D}(S_\lambda) \subsetneq \mathcal{D}(S_\lambda^*)$ (because, as shown in the last paragraph of the proof of Theorem 4.1, $\mathcal{D}(S_\lambda^*) \not\subseteq \mathcal{D}(S_\lambda)$ whenever $(\sum_{w \in \text{Chi}(v)} \lambda_w^2) \lambda_v^2 = 1$ for all $v \in V$).

4. FURTHER RESULTS AND REMARKS

Modifying the proof of Theorem 3.1, we can obtain the following.

Theorem 4.1. *If \mathcal{T} is a directed tree such that $\text{card}(\text{Chi}(u)) = \aleph_0$ for every $u \in V$, then there exists a family $\lambda = \{\lambda_v\}_{v \in V^\circ}$ of nonzero complex numbers such that S_λ is injective and densely defined, $\mathcal{D}(S_\lambda^2) = \{0\}$, $\mathcal{D}(S_\lambda) \not\subseteq \mathcal{D}(S_\lambda^*)$ and $\mathcal{D}(S_\lambda^*) \not\subseteq \mathcal{D}(S_\lambda)$.*

Proof. We begin by proving that the assertion (‡) is still valid if (3.4) is replaced by the following condition:

$$(4.1) \quad \sup_{v \in \text{Chi}(u)} \sum_{w \in \text{Chi}(v)} \frac{\lambda_w^4}{1 + \lambda_w^2} = \infty.$$

That this replacement is possible may be justified as follows. First, we find a family $\{\lambda_v\}_{v \in \text{Chi}(u)} \subseteq (0, \infty)$ such that $(\sum_{v \in \text{Chi}(u)} \lambda_v^2) \vartheta = 1$. Then evidently $\sup_{v \in \text{Chi}(u)} 1/\lambda_v^2 = \infty$, which together with [9, Proposition 2.1.2] and the fact that

for every real number $\alpha > 0$, there exists a sequence $\{\lambda_n\}_{n=1}^\infty \subseteq (0, \infty)$ such that $|\lambda_1 - \alpha| < 1$ and $\sum_{n=1}^\infty \lambda_n^2 = \alpha^2$,

enables us to construct a family $\{\lambda_w\}_{w \in \text{Chi}^{(2)}(u)} \subseteq (0, \infty)$ such that $\sum_{w \in \text{Chi}(v)} \lambda_w^2 = 1/\lambda_v^2$ for every $v \in \text{Chi}(u)$, and $\sup_{w \in \text{Chi}^{(2)}(u)} \lambda_w^2 = \infty$. This implies (4.1). Next, arguing as in the proof of (‡), we get a family $\{\lambda_v\}_{v \in \text{Des}(u)} \subseteq (0, \infty)$ which satisfies (3.3) for every $v \in \text{Des}(u)$, (3.2) and (4.1).

A reasoning similar to that in the proof of implication (i) \Rightarrow (ii) of Theorem 3.1 shows that there exists a family $\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq (0, \infty)$ such that (4.1) holds for at least one $u \in V$, say $u = u_1$, and $(\sum_{w \in \text{Chi}(v)} \lambda_w^2) \lambda_v^2 = 1$ for every $v \in V$ (with $\lambda_{\text{root}} := 1$ if \mathcal{T} has a root). This implies that S_λ is injective, $\overline{\mathcal{D}(S_\lambda)} = \ell^2(V)$, $\mathcal{D}(S_\lambda^2) = \{0\}$ and

$$(4.2) \quad \|S_\lambda e_v\|^2 = 1/\lambda_v^2, \quad v \in V^\circ,$$

which together with (4.1) leads to $\sup_{v \in \text{Chi}(u_1)} \sum_{w \in \text{Chi}(v)} \frac{\lambda_w^2}{1 + \|S_\lambda e_w\|^2} = \infty$. By applying [9, Theorem 4.1.1], we deduce that $\mathcal{D}(S_\lambda) \not\subseteq \mathcal{D}(S_\lambda^*)$.

Since for every $u \in V$ equation (4.2) holds for all $v \in \text{Chi}(u)$ and $\sum_{v \in \text{Chi}(u)} \lambda_v^2 < \infty$, we deduce that the function $\phi_u: \text{Chi}(u) \ni v \mapsto \|S_\lambda e_v\| \in \mathbb{C}$ is unbounded, and thus the operator M_u of multiplication by ϕ_u in $\ell^2(\text{Chi}(u))$ is unbounded (note that the function $\lambda^u: \text{Chi}(u) \ni v \mapsto \lambda_v \in \mathbb{C}$ does not belong to $\mathcal{D}(M_u)$, and so the definition [9, (4.2.2)] makes no sense). By applying [9, Theorem 4.2.2], we conclude that $\mathcal{D}(S_\lambda^*) \not\subseteq \mathcal{D}(S_\lambda)$. This completes the proof. \square

Remark 4.2. It is worth pointing out that if \mathcal{T} is a directed tree such that $\text{card}(\text{Chi}(u)) = \aleph_0$ for every $u \in V$, S_λ is a densely defined weighted shift on \mathcal{T} with nonzero weights $\lambda = \{\lambda_v\}_{v \in V^\circ}$ such that $\mathcal{D}(S_\lambda^2) = \{0\}$ (cf. Theorem 3.1) and $v_0 \in V^\circ$, then the weighted shift $S_{\tilde{\lambda}}$ on \mathcal{T} with nonzero weights $\tilde{\lambda} = \{\tilde{\lambda}_v\}_{v \in V^\circ}$ given by

$$\tilde{\lambda}_v = \begin{cases} \lambda_v & \text{for } v \neq v_0, \\ \sqrt{1 + \|S_\lambda e_v\|^2} & \text{for } v = v_0, \end{cases}$$

has the following properties: $\mathcal{D}(S_\lambda) = \mathcal{D}(S_{\tilde{\lambda}})$ (use [9, Proposition 3.1.3]), $\mathcal{D}(S_\lambda^*) = \mathcal{D}(S_{\tilde{\lambda}}^*)$ (use [9, Proposition 3.4.1]), $S_{\tilde{\lambda}}$ is not hyponormal (use Theorem 2.1) and $\mathcal{D}(S_{\tilde{\lambda}}^2) = \{0\}$ (use (\dagger)). Hence, if S_λ is as in the proof of implication (i) \Rightarrow (ii) of Theorem 3.1, then by Remark 3.2 we have $\mathcal{D}(S_{\tilde{\lambda}}) \subsetneq \mathcal{D}(S_{\tilde{\lambda}}^*)$.

We conclude the paper with a brief discussion of the case of composition operators in L^2 -spaces. The following theorem is a direct consequence of Theorem 3.1, Remark 3.2, [9, Theorem 3.2.1] and [10, Lemma 4.3.1].

Theorem 4.3. *There exists a hyponormal composition operator C in an L^2 -space over a σ -finite measure space such that $\mathcal{D}(C^2) = \{0\}$ and $\mathcal{D}(C) \subsetneq \mathcal{D}(C^*)$.*

Note also that, by Theorem 4.1, [9, Theorem 3.2.1] and [10, Lemma 4.3.1], there exists an injective densely defined composition operator C in an L^2 -space over a σ -finite measure space such that $\mathcal{D}(C^2) = \{0\}$, $\mathcal{D}(C) \not\subseteq \mathcal{D}(C^*)$ and $\mathcal{D}(C^*) \not\subseteq \mathcal{D}(C)$.

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